AD-A068 899 WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER F/G 12/1
A STRAIGHTFORWARD GENERALIZATION OF DILIBERTO AND STRAUS' ALGOR--ETC(U)
JAN 79 N RICHTER-DYN DAAG29-75-C-0024
UNCLASSIFIED MRC-TSR-1916 NL

1 OF 1
ADA
068899

END
DATE
FILMED
6-79
DDC

MRC Technical Summary Report # 1916

A STRAIGHTFORWARD GENERALIZATION OF DILIBERTO AND STRAUS' ALGORITHM DOES NOT WORK

Nira Richter-Dyn

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

January 1979

Received December 22, 1978

Sponsored by

U.S. Army Research Office
P.O. Box 12211
Research Triangle Park
North Carolina 27709

UNIVERSITY OF WISCONSIN - MADISON
MATHEMATICS RESEARCH CENTER

6

A STRAIGHTFORWARD GENERALIZATION OF DILIBERTO AND STRAUS' ALGORITHM DOES NOT WORK.

10 Nira/Richter-Dyn

9 Technical ~~Summary Report~~, 1916
~~January 1979~~

12 8p.

11 Jan 79

ABSTRACT

An algorithm for best approximating in the sup-norm a function $f \in C[0,1]^2$ by functions from tensor-product spaces of the form $\pi_k \otimes C[0,1] \oplus C[0,1] \otimes \pi_\ell$, is considered. For the case $k = \ell = 0$ the Diliberto and Straus algorithm is known to converge. A straightforward generalization of this algorithm to general k,ℓ is formulated, and an example is constructed demonstrating that this algorithm does not converge for $k^2 + \ell^2 > 0$.

14 MRC-TSR-1916

AMS(MOS) Subject Classifications: 41A50, 41A63

Key Words: Algorithm, Tensor-Product Spaces,
Best Approximation by Polynomials

Work Unit Number 6 - Spline Functions and Approximation Theory

15

Sponsored by the United States Army under Contract No. DAAG29-75-C-~~00~~24.

SIGNIFICANCE AND EXPLANATION

Often it is desirable to approximate a given function as closely as possible by a member of a class of functions that are simpler to evaluate.

For a general continuous function of two variables $f(x,y)$ a best approximating function of the simpler form $h(y) + g(x)$ can be computed by the algorithm of Diliberto and Straus. Since such an approximation can be quite far from the approximated function, a better approximation of the form $\sum_{i=0}^{k} h_i(y)x^i + \sum_{j=0}^{\ell} g_i(x)y^i$ is considered. One way to try to construct such an approximation is to generalize the Diliberto and Straus algorithm to this more general setting. The generalized algorithm is simple in the sense that only one-dimensional best approximations by polynomials have to be computed. In this note it is shown by a simple example, that this "natural" generalization cannot be expected to converge, and therefore other methods should be developed .

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

A STRAIGHTFORWARD GENERALIZATION OF DILIBERTO AND STRAUS' ALGORITHM DOES NOT WORK

Nira Richter-Dyn

The algorithm of Diliberto and Straus for approximating a bivariate function by a sum of univariate ones proposed in 1951 [1], has been recently investigated in several works [2], [3], [4], where convergence and various properties of the algorithm are studied.

The algorithm, designed for computing the best approximation to $f \in C[0,1]^2$ in the sup-norm from the space

(1) $$M = \{\phi \mid \phi(x,y) \in C[0,1]^2,\ \phi(x,y) = h(y) + g(x)\},$$

is of the following form:

$$f_0(x,y) = f(x,y)$$

(2) $$f_{2n+1}(x,y) = f_{2n}(x,y) - \frac{1}{2}\left[\max_{0 \le \xi \le 1} f_{2n}(\xi,y) + \min_{0 \le \xi \le 1} f_{2n}(\xi,y)\right], \quad n = 0,1,\dots,$$

$$f_{2n+2}(x,y) = f_{2n+1}(x,y) - \frac{1}{2}\left[\max_{0 \le \eta \le 1} f_{2n+1}(x,\eta) + \min_{0 \le \eta \le 1} f_{2n+1}(x,\eta)\right], \quad n = 0,1,\dots .$$

It is proved in [1], [3], [4] that $\lim_{n\to\infty}\|f_n\| = \inf_{\phi\in M}\|f-\phi\|$, although the rate of convergence might be extremely slow [2]. Algorithm (2) can be interpreted as a sequence of repeated applications of the operator of one dimensional best approximation by constants to $f(x,y)$, regarded alternately as a function of x and as a function of y. More specifically, let J_x be the operator associating with $f(x,y) \in C[0,1]^2$ the function $(J_x f)(y) \in C[0,1]$, with $(J_x f)(y_0)$ the constant of best approximation to $f(x,y_0)$ in the sup-norm on $[0,1]$, and let J_y be defined similarly with the roles of x,y interchanged. Then (2) can be rewritten as

(3) $$f_0=f,\ f_{2n+1}=f_{2n}-J_x f_{2n},\ f_{2n+2}=f_{2n+1}-J_y f_{2n+1},\quad n=0,1,2,\dots .$$

Sponsored by the United States Army under Contract No. DAAG29-75-C-0024.

This formulation suggests a straightforward generalization of algorithm (3), namely best approximating $f(x,y)$ alternately in the x and y directions by polynomials of degree k and ℓ respectively, in order to obtain a best approximation to $f(x,y)$ from the tensor-product space

(4) $$M_{k,\ell} = \{\phi(x,y)\,|\,\phi(x,y) \in C[0,1]^2, \phi(x,y) = \sum_{j=0}^{k} h_j(y)x^j + \sum_{j=0}^{\ell} g_j(x)y^j\} =$$

$$= \pi_k \otimes C[0,1] \oplus C[0,1] \otimes \pi_\ell \,.$$

(π_k denotes the space of all univariate polynomials of degree $\leq k$.) With this notation the subspace M in (1) is the tensor-product space $M_{0,0}$. The generalization of algorithm (3) to this more general setting is

(5) $$f_0=f,\ f_{2n+1}=f_{2n}-J_x^{(k)}f_{2n},\ f_{2n+2}=f_{2n+1}-J_y^{(\ell)}f_{2n+1},\quad n=0,1,2,\ldots\ ,$$

where $(J_x^{(k)}f)(x,y_0) = \sum_{j=0}^{k} h_j(y_0)x^j$ is the polynomial of best approximation to $f(x,y_0)$ in the sup-norm on $[0,1]$ from π_k, and where $(J_y^{(\ell)}f)(x_0,y)$ is similarly defined.

In the following we present a simple example demonstrating that algorithm (5) for general k,ℓ cannot be expected to converge to a best approximation to $f_0(x,y)$. We construct a function $f(x,y)$ such that $\|f\| > \inf_{\phi\in M_{0,1}} \|f-\phi\|$, while the functions $\{f_n\}$ generated from it by (5) with $k=0,\ell=1$ satisfy $\|f_n\| = \|f\|$ for all n.

Consider $f(x,y)$ $C[0,1]^2$ subject to the following conditions:

(6) $$\begin{aligned} &f(\tfrac{i}{4},\tfrac{j}{6}) = (-1)^{i+j}\,, && j=2i,2i+1,2i+2,\ i=0,1,2 \\ &f(\tfrac{3}{4},\tfrac{j}{6}) = (-1)^{j+1}\,, && j=0,5,6 \\ &f(1,\tfrac{2j+1}{6}) = (-1)^{j}\,, && j=0,1,2 \\ &|f(x,y)| < 1 && \text{elsewhere in } [0,1]^2\,. \end{aligned}$$

As can be easily observed

$$(J_x^{(0)}f)(x,\tfrac{i}{6}) = 0\,,\ i=0,1,\ldots,6 \quad\text{and}\quad (J_y^{(1)}f)(\tfrac{i}{4},y)=0,\ i=0,1,2,3,4\,,$$

and both $f-J_x^{(0)}f$ and $f-J_y^{(1)}f$ satisfy (6). Thus algorithm (5) with $k=0,\ell=1$ generates a sequence $\{f_n\}$ of functions satisfying (6) whenever f_0 satisfies (6), and therefore $\|f_n\| = 1$ for all $n \geq 0$.

In order to verify that $\|f\| > \inf_{\phi\in M_{0,1}} \|f-\phi\|$, it is sufficient to show that there does not exist a bounded linear functional $\mu \in (C[0,1]^2)'$, $\mu \neq 0$, such that

(7) $\langle \phi,\mu \rangle = 0$ for all $\phi \in M_{0,1}$,

(8) $\langle f,\mu \rangle = \|\mu\|$.

Indeed any $\mu\neq 0$ with property (8) is necessarily of the form

(9) $\langle \zeta,\mu \rangle = \sum_{j=0}^{r} a_j\zeta(x_j,y_j)$, $\zeta \in C[0,1]^2$, with $r > 0$, $a_j f(x_j,y_j) = |a_j|$, $j=0,\ldots,r$,

namely a linear combination of function values at extremal points of f . Moreover condition (7) implies that μ can be written as a linear combination of first differences in the x direction so as to vanish on all functions of the form $h(y)$, and as a linear combination of second order divided differences in the y direction, so as to vanish on all functions of the form $g_0(x) + g_1(x)y$.

These characteristics of μ are consistent with the special structure of the 15 extremal points of f , as given in (6), only if $r=14$ in (9). Then μ can be written as

(10) $\langle \zeta,\mu \rangle = \sum_{i=0}^{4} c_i[\ \ ]_i\zeta$,

where $[\ \ ]_i\zeta$ denotes the second order divided difference of $\zeta(\frac{i}{4},y)$ at the extremal points of f with $x=\frac{i}{4}$. The sum (10) can be rewritten as a linear combination of first differences in the x direction only if $c_0,\ldots,c_4$ satisfy the following system of linear equations:

$$c_0=c_1=c_2,\quad c_2=\frac{c_3}{3},\quad c_1=\frac{c_4}{4},\quad c_0=\frac{c_3}{15},\quad c_2=\frac{2}{5}c_3+\frac{c_4}{4},$$

which admits only the trivial solution.

Acknowledgement: The author wishes to thank Professor Carl de Boor for valuable discussions, and for his suggestions that lead to a clearer representation of the material in this note.

REFERENCES

1. Diliberto, S. P. and Straus, E.g., "On the approximation of a function of several variables by a sum of functions of fewer variables", Pacific J. Math. 1 (1951), 195-210.
2. von Golitschek, M. and Cheney, E. W., "On the algorithm of Diliberto and Straus for approximating bivariate functions by univariate ones", Report CNA-141, Center of Numerical Analysis, University of Texas at Austin, August 1978.
3. Kelley, C.T., "A note on the approximation of functions of several variables by sums of functions of one variable", Report No. 1873, Mathematics Research Center, University of Wisconsin-Madison, August 1978.
4. Light, W. A. and CHeney, E. W., "On the approximation of a bivariate function by the sum of univariate functions", Report CNA-140, Center for numerical Analysis, University of Texas at Austin, August 1978.

NRD/db

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER
1916

2. GOVT ACCESSION NO.

3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle)
A STRAIGHTFORWARD GENERALIZATION OF DILIBERTO AND STRAUS' ALGORITHM DOES NOT WORK

5. TYPE OF REPORT & PERIOD COVERED
Summary Report - no specific reporting period

6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s)
Nira Richter-Dyn

8. CONTRACT OR GRANT NUMBER(s)
DAAG29-75-C-0024

9. PERFORMING ORGANIZATION NAME AND ADDRESS
Mathematics Research Center, University of Wisconsin
610 Walnut Street
Madison, Wisconsin 53706

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
#6 - Spline Functions and Approximation Theory

11. CONTROLLING OFFICE NAME AND ADDRESS
U. S. Army Research Office
P.O. Box 12211
Research Triangle Park, North Carolina 27709

12. REPORT DATE
January 1979

13. NUMBER OF PAGES
4

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)

15. SECURITY CLASS. (of this report)
UNCLASSIFIED

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)
Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)
Algorithm, Tensor-Product Spaces, Best Approximation by Polynomials

pi sub k

an element of

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

An algorithm for best approximating in the sup-norm a function $f \in C([0,1]^2)$ squared by functions from tensor-product spaces of the form $\pi_k \otimes C[0,1] \oplus C[0,1] \otimes \pi_\ell$, pi sub l is considered. For the case, $k=\ell=0$ the Diliberto and Straus algorithm is known to converge. A straightforward generalization of this algorithm to general k,ℓ is formulated, and an example is constructed demonstrating that this algorithm does not converge for $k^2+\ell^2 > 0$.

k-squared ℓ-squared x +

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED
SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)